On the Extension of the Uniform Rule to More Than One Commodity: Existence and Maximality Results

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Abstract

We study the problem of allocating a bundle of non-disposable, infinitely divisible commodities among a group of agents with multi-dimensional single-peaked preferences, working with several definitions which generalize peak-separable single-peakedness, a concept that has been considered by previous writers. We establish conditions under which one-dimensional rules, most notably, the uniform rule, can be extended to that setting and conditions under which single-valuedness is preserved. Also, weakening one of the two requirements of peak-separable single-peakedness at a time, we identify four maximal domains of preferences for *commodity-wise same-sidedness*, *no-envy* (and *equal treatment of equals in physical terms*) and *strategy-proofness*.

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Key Words: Commodity-wise single-peakedness; commodity-wise uniform rule; maximal domain

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1 Introduction

The problem of allocating a social endowment of an infinitely divisible good among agents with single-peaked preferences has been the object of considerable attention. A number of rules can be defined for that purpose, but a particular rule has been found to be central from the viewpoint of incentives as well as from various normative viewpoints. Thus rule, known as the "uniform rule" (Bénassy, 1982), has indeed been characterized in many different ways, starting with Sprumont (1991). (See Thomson (2017) for a survey of these results.)

We consider a multi-commodity generalization of the problem, for which a natural extension of the notion of single-peakedness has been proposed (Amorós, 1999, 2002; Anno and Sasaki, 2009; Adachi, 2010; Morimoto, Serizawa, and Ching, 2013). According to this definition, (i) keeping fixed the consumptions of all commodities except one, single-peakedness in the usual sense holds with respect to the last commodity; moreover, (ii) the maximizer of this induced relation is independent of the consumptions of the other commodities. We call such preferences peak-separable single-peaked.[1] Also, we refer to requirement (i) as condition SP (for "single-peaked") and to requirement (ii) as condition C (for "constant peak"). The various rules that have been introduced in the one-commodity case can be easily extended to this domain. It suffices to apply them commodity by commodity. Characterizations of an extension of the uniform rule obtained in this manner, on the basis of a list of axioms parallel to the list shown by Sprumont (1991) and Ching (1994) to lead to it in the one-commodity case, have also been established (Amorós, 2002; Adachi, 2010; Morimoto, Serizawa, and Ching, 2013).

Our goal is to explore how far one can go from this domain without compromising the existence of rules satisfying the strategic properties that the commodity-wise uniform rule enjoys. Because commodity-wise peak-separable single-peakedness is the conjunction of two requirements, we break down our query into two parts, each corresponding to dropping one of the two requirements. A preliminary question we need to address, however, is whether "the one-commodity uniform rule" can be meaningfully adapted to the domains enlarged in this manner, and, when a "commodity-wise uniform solution" can be defined, whether this solution inherits the properties of its one-dimensional origin.

First, we show that if, with condition C still in place, condition SP is weakened so as to allow for what we call "one-dimensional weakly single-plateaued preferences", a commodity-wise uniform rule can indeed be defined and that this mapping is "essentially single-valued" (that is, single-valued in welfare terms). It violates "efficiency" but it satisfies "commodity-wise same-sidedness", the requirement that for each commodity separately, either each agent is assigned at most as much as his peak amount of that commodity, or each agent is assigned at least as much as his peak amount,

[1] Serizawa refers to them as "cross-shaped".

a condition that is necessary for efficiency (and equivalent to efficiency in the one-commodity case). It also satisfies "equal treatment of equals in physical terms" (and therefore, in welfare terms), the requirement that two agents with the same preferences be assigned equal bundles (and bundles that they find indifferent according to their common preferences), "no-envy", the requirement that each agent find his assignment at least as desirable as anyone else's assignment, and "strategy-proofness", the requirement that in the direct revelation game form associated with the rule, no agent ever find it beneficial to misrepresent his preferences.

On the other hand, if only condition C is dropped, we obtain commodity-wise single-peaked preferences. On this extended domain, the commodity-wise uniform solution is well defined (Theorem 1), but it is not single-valued (nor even essentially single-valued). However, we identify a sufficient condition on preferences for single-valuedness to hold. This condition is a bound on the absolute value of the slopes of the loci of maximizers along affine sets parallel to axes (Theorem 2). Unfortunately, even though single-valuedness is recovered, many interesting properties satisfied by the one-dimensional uniform rule are lost. Notably, the commodity-wise uniform rule violates efficiency, no-envy, the equal-division lower bound, and strategy-proofness.

The next natural question is whether these failures are unique to the commodity-wise uniform rule. And if not, how much can the domain of peak-separable single-peaked preferences be enlarged without the existence of a rule satisfying a list of axioms of our choice being compromised? Again, with conditions SP and C in mind, we pursue this question in two directions. We show that if condition SP is maintained, then the domain of peak-separable single-peaked preferences is a maximal domain on which commodity-wise same-sidedness, no-envy, and strategy-proofness are compatible (Theorem 4). On the other hand, if condition C is kept, then the domain of self-explanatory plateau-separable weakly single-plateaued preferences is a maximal domain on which these properties are compatible (Theorem 6). For two agents, even stronger conclusions can be reached (Theorems 3 and 5).

Our results imply that the domain on which Amorós (2002), Adachi (2010), and Morimoto, Serizawa, and Ching (2013) characterize the commodity-wise uniform rule is the largest domain on which their axioms are compatible. Our maximality results (Theorems 3 and 4), which pertain to weakening condition C, but not SP, show that the invariance of peak amounts with respect to the consumption level of all but one commodity is critical in ensuring the existence of a rule satisfying their axioms in that even the slightest departure from it yields an impossibility.

The remainder of the paper is organized as follows. In Section 2 we set up the model and define various classes of preferences and axioms. In Section 3 we generalize the one-dimensional uniform rule for our model and study its properties. In Section 4 we present our maximal domain results, and in Section 5 we prove the theorems.

2 The Model

Let $L \equiv \{1, \ldots, \bar{\ell}\}$ be a set of (infinitely divisible) commodities, and $N \equiv \{1, ..., n\}$ be a set of agents. For each $i \in N$, agent i's consumption space is $\mathbb{R}^L_+$, and agent i is equipped with a complete, transitive, and continuous **preference relation** R_i over $\mathbb{R}^L_+$. Let P_i and I_i denote the strict preference and indifference relations associated with R_i. Let $\boldsymbol{\mathcal{R}}$ be our generic notation for a class of such preferences. Let $R \equiv (R_i)_{i \in N} \in \mathcal{R}^N$ be a profile of preferences.

There is a **social endowment** $\Omega \equiv (\Omega_1, ..., \Omega_{\bar{\ell}}) \in \mathbb{R}^L_+$ to be fully allocated among the members of N. Altogether, an **economy** is a pair $(R, \Omega) \in \mathcal{R}^N \times \mathbb{R}^\ell_+$. A (feasible) **allocation** for (R, Ω) is a vector $x \equiv (x_1, ..., x_n) \in \mathbb{R}^{LN}_+$, where for each $i \in N$, $x_i \equiv (x_{i1}, ..., x_{i\bar{\ell}}) \in \mathbb{R}^L_+$ is agent i's assignment, and for each $\ell \in L$, $\sum_{i \in N} x_{i\ell} = \Omega_\ell$. Let X be the set of all allocations for (R, Ω). A **rule** defined on $\mathcal{R}^N \times \mathbb{R}^L_+$ is a mapping that associates with each economy (R, Ω) in its domain an allocation for (R, Ω). Our generic notation for a rule is the letter φ.

2.1 Preferences

Next, we introduce various classes of preferences, first in the one-commodity case (i.e., when $\bar{\ell} = 1$). A preference relation $R_0 \in \mathcal{R}$ is **one-dimensional single-plateaued** if there are $\underline{p}(R_0), \bar{p}(R_0) \in \mathbb{R}_+$ with $\underline{p}(R_0) \leq \bar{p}(R_0)$, such that (i) for each pair $x_0, x'_0 \in \mathbb{R}_+$ such that either $x'_0 < x_0 \leq \underline{p}(R_0)$ or $\bar{p}(R_0) \leq x_0 < x'_0$, we have $x_0 \, P_0 \, x'_0$; and (ii) for each pair $x_0, x'_0 \in \mathbb{R}_+$ such that $\underline{p}(R_0) \leq x_0 \leq x'_0 \leq \bar{p}(R_0)$, we have $x_0 \, I_0 \, x'_0$. Also, a preference relation $R_0 \in \mathcal{R}$ is **one-dimensional weakly single-plateaued** if there are $\underline{p}(R_0), \bar{p}(R_0) \in \mathbb{R}_+$ with $\underline{p}(R_0) \leq \bar{p}(R_0)$, such that (i) for each pair $x_0, x'_0 \in \mathbb{R}_+$ such that either $x'_0 < x_0 \leq \underline{p}(R_0)$ or $\bar{p}(R_0) \leq x_0 < x'_0$, we have $x_0 \, R_0 \, x'_0$; and (ii) for each pair $x_0, x'_0 \in \mathbb{R}_+$ such that $\underline{p}(R_0) \leq x_0 \leq x'_0 \leq \bar{p}(R_0)$, we have $x_0 \, I_0 \, x'_0$; (iii) $[\underline{p}(R_0), \bar{p}(R_0)]$ is the set of maximizers of R_0 on $\mathbb{R}_+$. The interval $p(R_0) \equiv [\underline{p}(R_0), \bar{p}(R_0)]$ is called the **plateau** of R_0. Starting from these two definitions, if $\underline{p}(R_0) = \bar{p}(R_0)$, we say that R_0 is **one-dimensional single-peaked** and **one-dimensional weakly single-peaked**, respectively, and we call $p(R_0)$ the **peak amount** of R_0. In the latter case, we abuse notation slightly and treat $p(R_0)$ as a point in $\mathbb{R}_+$. Let $\boldsymbol{\mathcal{R}_{1.sg.pk}}$ be the class of one-dimensional single-peaked preferences.

Weakly single-plateaued preferences are more general than single-plateaued preferences in that non-degenerate intervals of indifferent points (ledges) are permitted to the left or right of plateaus: A numerical representation of a single-plateaued relation is strictly monotonic increasing to the left of the plateau and strictly monotonic decreasing to the right of the plateau; a representation of a weakly single-plateaued relation may only be weakly monotonic in these intervals.

We now turn to the multi-commodity case, when $\bar{\ell} \geq 2$. Multi-dimensional single-peakedness can be defined in several different ways, reviewed in Thomson (2010). We start with the definition

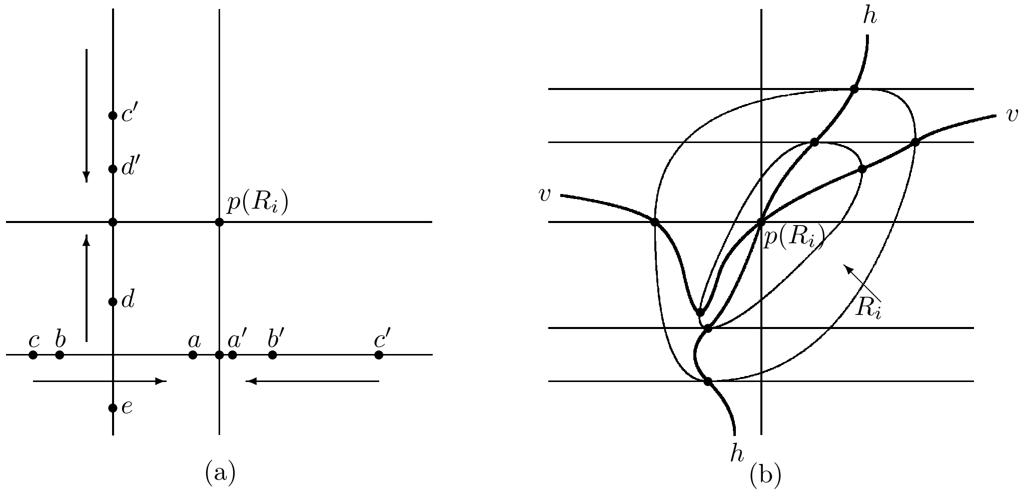


Figure 1: **Generalizing single-peakedness of a preference relation to a two-dimensional Euclidean space: "commodity-wise single-peakedness".** (a) Along each horizontal line and along each vertical line, preferences are one-dimensional single-peaked. Direction of increasing preferences is indicated by the arrows. Indifferent points are denoted by the same letter, with and without a prime. (b) The curve v-v is the locus of the maximizer of the relation along vertical lines and the curve h-h is the locus of the maximizer of the relation along horizontal lines. Upper contour sets need not be convex although they are in this example.

that has been studied the most.

Peak-separable single-peakedness: For each $\ell \in L$, there is $p_\ell(R_0) \in \mathbb{R}_+^{\{\ell\}}$ such that for each $x_{-\ell} \in \mathbb{R}_+^{L\setminus\{\ell\}}$, the restriction of R_0 to the set $\{(x_\ell, x_{-\ell})\colon x_\ell \in \mathbb{R}_+^{\{\ell\}}\}$ is one-dimensional single-peaked with its peak amount equal to $p_\ell(R_0)$.

Let $\boldsymbol{\mathcal{R}_{pk.sep-sg.pk}}$ denote the class of these preferences, and for each $R_0 \in \mathcal{R}_{pk.sep-sg.pk}$, let $p(R_0) \equiv (p_\ell(R_0))_{\ell\in L}$. The requirement of peak-separability single-peakedness has two parts: (i) the restriction of R_0 to each affine set $\{(x_\ell, x_{-\ell}) : x_\ell \in \mathbb{R}_+^{\{\ell\}}\}$ is one-dimensional single-peaked; and (ii) as we vary $x_{-\ell} \in \mathbb{R}_+^{L\setminus\{\ell\}}$, the peak amount of the relation restricted in this manner is constant at $p_\ell(R_0)$. We refer to condition (i) as SP (for "single-peaked") and to condition (ii) as C (for "constant peak"). Condition SP pertains to the commodity-wise property of preferences whereas condition C pertains to the peak–or more generally, plateau–property.

Since peak-separable single-peakedness is defined as the conjunction of two conditions, we may inquire about the kind of preferences that are additionally permitted if we relax only one of them. First, we relax C and maintain SP: keeping fixed an agent's consumption level of all but one commodity, his welfare is represented by a strictly quasi-concave, or equivalently single-peaked, function of the last commodity; however, the consumption of the last commodity that yields the maxima (or peak amounts) of those functions may change as we vary the consumption level of the other commodities.

Commodity-wise single-peakedness: For each $\ell \in L$ and each $x_{-\ell} \in \mathbb{R}_+^{L\setminus\{\ell\}}$, the restriction of R_0 to the set $\{(x_\ell, x_{-\ell})\colon x_\ell \in \mathbb{R}_+^{\{\ell\}}\}$ is one-dimensional single-peaked.

Let $\boldsymbol{\mathcal{R}_{sg.pk}}$ denote the class of all commodity-wise single-peaked preferences. If $R_0 \in \mathcal{R}_{sg.pk}$, then to each $\ell \in L$, there corresponds a locus of peak amounts, namely the graph of the function $p_\ell(\cdot|R_0)\colon \mathbb{R}_+^{L\setminus\{\ell\}} \to \mathbb{R}_+^{\{\ell\}}$. In the case of two commodities, there are two such loci, a locus $v-v$ of peak amounts on vertical lines, and a locus $h-h$ of peaks on horizontal lines. Let $\boldsymbol{v(\cdot|R_0)\colon} \mathbb{R}_+^{\boldsymbol{\{1\}}} \to \mathbb{R}_+$ and $\boldsymbol{h(\cdot|R_0)\colon} \mathbb{R}_+^{\boldsymbol{\{2\}}} \to \mathbb{R}_+$ be parameterizations of those loci. Since $\mathcal{R}_{pk.sep-sg.pk} \subsetneq \mathcal{R}_{sg.pk}$, we treat $R_0 \in \mathcal{R}_{pk.sep-sg.pk}$ as a preference relation in $\mathcal{R}_{sg.pk}$, so that for each $\ell \in L$ and each $x_{-\ell} \in \mathbb{R}_+^{L\setminus\{\ell\}}$, $p_\ell(x_{-\ell}|R_0) = p_\ell(R_0)$.

Figure 1b shows the loci h-h and v-v for an example. These loci can have very general shapes. However, they can only cross at the global maximizer of the relation. Also, if preferences are smooth, they cannot meet anywhere else. If kinks in indifference curves are allowed, they can meet at other points. In fact, there are preferences for which the two loci coincide (as shown in Figure 5 below, whose main purpose is to illustrate a different point). Figure 1b shows an example in which the upper contour sets are convex but they need not be. Indeed, this definition does not imply convexity of preferences.

Next, we generalize peak-separable single-peakedness by relaxing condition SP while maintaining C: keeping fixed an agent's consumption of all commodities but but one, his welfare is represented by a single-plateaued function of the last commodity; moreover, the consumption levels of the last commodity that yields the maxima (or plateaus) of those functions is independent of the consumption level of the other commodities.

Plateau-separable single-plateauedness: For each $\ell \in L$, there are $\underline{p}_\ell(R_0), \bar{p}_\ell(R_0) \in \mathbb{R}_+^{\{\ell\}}$, with $\underline{p}_\ell(R_0) \leq \bar{p}_\ell(R_0)$, such that for each $x_{-\ell} \in \mathbb{R}_+^{L\setminus\{\ell\}}$, the restriction of R_0 to the set $\{(x_\ell, x_{-\ell}) : x_\ell \in \mathbb{R}_+^{\{\ell\}}\}$ is one-dimensional single-plateaued with its plateau equal to $[\underline{p}_\ell(R_0), \bar{p}_\ell(R_0)]$.

On each affine set $\{(x_\ell, x_{-\ell}) : x_\ell \in \mathbb{R}_+^{\{\ell\}}\}$, plateau-separable single-plateauedness allows indifference only on the plateau; to the left and right of the plateau, welfare is increasing and decreasing, respectively. Weakening strict monotonicity to weak monotonicity yields plateau-separable weak single-plateauedness.

Plateau-separable weak single-plateauedness: For each $\ell \in L$, there are $\underline{p}_\ell(R_0)$, $\bar{p}_\ell(R_0) \in \mathbb{R}_+^{\{\ell\}}$, with $\underline{p}_\ell(R_0) \leq \bar{p}_\ell(R_0)$, such that for each $x_{-\ell} \in \mathbb{R}_+^{L\setminus\{\ell\}}$, the restriction of R_0 to the set $\{(x_\ell, x_{-\ell})\colon x_\ell \in \mathbb{R}_+^{\{\ell\}}\}$ is one-dimensional weakly single-plateaued with its plateau equal to $[\underline{p}_\ell(R_0), \bar{p}_\ell(R_0)]$.

Let $\boldsymbol{\mathcal{R}_{pl.sep-w.sg.pl}}$ denote the class of plateau-separable weakly single-plateaued preferences. Such preferences retain the following commodity-wise property of peak-separable single-peakedness: for each $\ell \in K$ and each $x_{-\ell} \in \mathbb{R}^{K\setminus\ell}$, upper contour sets of the restriction of R_0 to affine sets $\{(x_\ell, x_{-\ell}) : x_\ell \in \mathbb{R}_+^{\{\ell\}}\}$ are convex. Thus, to further extend plateau-separable weak single-plateauedness, we give up on commodity-wise convexity as well, and only require that plateaus be invariant with respect to the consumption level of $\bar{\ell}-1$ commodities.

Plateau-separability: For each $\ell \in L$, there are $\underline{p}_\ell(R_0)$, $\bar{p}_\ell(R_0) \in \mathbb{R}_+^{\{\ell\}}$, with $\underline{p}_\ell(R_0) \leq \bar{p}_\ell(R_0)$, such that for each $x_{-\ell} \in \mathbb{R}_+^{L\setminus\{\ell\}}$, $[\underline{p}_\ell(R_0), \bar{p}_\ell(R_0)]$ is the set of maximizers of the restriction of R_0 to the set $\{(x_\ell, x_{-\ell})\colon x_\ell \in \mathbb{R}_+^{\{\ell\}}\}$.

Denote by $\boldsymbol{\mathcal{R}_{pl.sep}}$ the class of plateau-separable preferences. For each $R_0 \in \mathcal{R}_{pl.sep}$ and each $\ell \in L$, let $p_\ell(R_0) \equiv [\underline{p}_\ell(R_0), \bar{p}_\ell(R_0)]$, and $p(R_0) \equiv \times_{\ell \in L} p_\ell(R_0)$. Figure 2 presents, in a schematic way, the classes of preferences introduced so far. On the horizontal axis we indicate the plateau (peak) property of preferences, and on the vertical axis the commodity-wise property. In this coordinate system, the greater the abscissa or ordinate, the weaker the corresponding property. For instance, weakening condition SP while keeping C amounts to moving vertically from peak-separable single-peakedness, and varying the degree of that weakening, we obtain, in succession, plateau-separable single-plateauedness, plateau-separable weak single-plateauedness, and plateau-separability.

We introduce further notation. For each $R_0 \in \mathcal{R}$ and each $\ell \in L$, let $\boldsymbol{\underline{p}_\ell(\cdot|R_0)} : \mathbb{R}_+^{L\setminus\{\ell\}} \rightarrow \mathbb{R}_+^{\{\ell\}} \cup \{\infty\}$ and $\boldsymbol{\bar{p}_\ell(\cdot|R_0)} : \mathbb{R}_+^{L\setminus\{\ell\}} \rightarrow \mathbb{R}_+^{\{\ell\}} \cup \{\infty\}$ be functions, with $\underline{p}_\ell(\cdot|R_0) \leq \bar{p}_\ell(\cdot|R_0)$, such that for each $x_{-\ell} \in \mathbb{R}_+^{L\setminus\{\ell\}}$, the interval $[\underline{p}_\ell(x_{-\ell}|R_0), \bar{p}_\ell(x_{-\ell}|R_0)]$ denotes the set of maximizers of R_0 in the one-dimensional set $\{(x_\ell, x_{-\ell})\colon x_\ell \in \mathbb{R}_+^{\{\ell\}}\}$. This notation is consistent, although with a slight abuse, with that we use for preferences in $\mathcal{R}_{sg.pk}$ and $\mathcal{R}_{pl.sep}$: If $R_0 \in \mathcal{R}_{sg.pk}$, then $\underline{p}_\ell(x_{-\ell}|R_0) = \bar{p}_\ell(x_{-\ell}|R_0) = p(x_{-\ell}|R_0)$; if $R_0 \in \mathcal{R}_{pl.sep}$, then $\underline{p}_\ell(x_{-\ell}|R_0) = \underline{p}_\ell(R_0)$ and $\bar{p}(x_{-\ell}|R_0) = \bar{p}_\ell(R_0)$.

To distinguish the model from its multi-commodity counterpart, we qualify it of "one-dimensional".

2.2 Axioms

In this section, we introduce the axioms. Let $\varphi\colon \mathcal{R}^N \times \mathbb{R}_+^L \rightarrow \mathbb{R}_+^{LN}$ be a rule.

Let $(R, \Omega) \in \mathcal{R}^N \times \mathbb{R}_+^L$ and let $x, y \in \mathbb{R}_+^{LN}$ be allocations for (R, Ω). Say that $\boldsymbol{x}$ **Pareto dominates** $\boldsymbol{y}$ if (i) for each $i \in N$, $x_i \, R_i \, y_i$, and (ii) there is $i \in N$ such that $x_i \, P_i \, y_i$. Our first axiom says that the rule should never choose an allocation that is Pareto dominated by some other allocation.

Efficiency: For each $(R, \Omega) \in \mathcal{R}^N \times \mathbb{R}_+^L$, there is no allocation that Pareto dominates $\varphi(R, \Omega)$.

In the one-commodity case, *efficiency* reduces to the following: an allocation is **same-sided** if (i) each agent's assignment is at most as large as his peak amount; or (ii) each agent's assignment is at least as large as his peak amount. For two or more commodities, we may apply this requirement commodity-wise, requiring *same-sidedness* for each commodity separately. The condition so obtained is necessary for *efficiency*, but not sufficient.

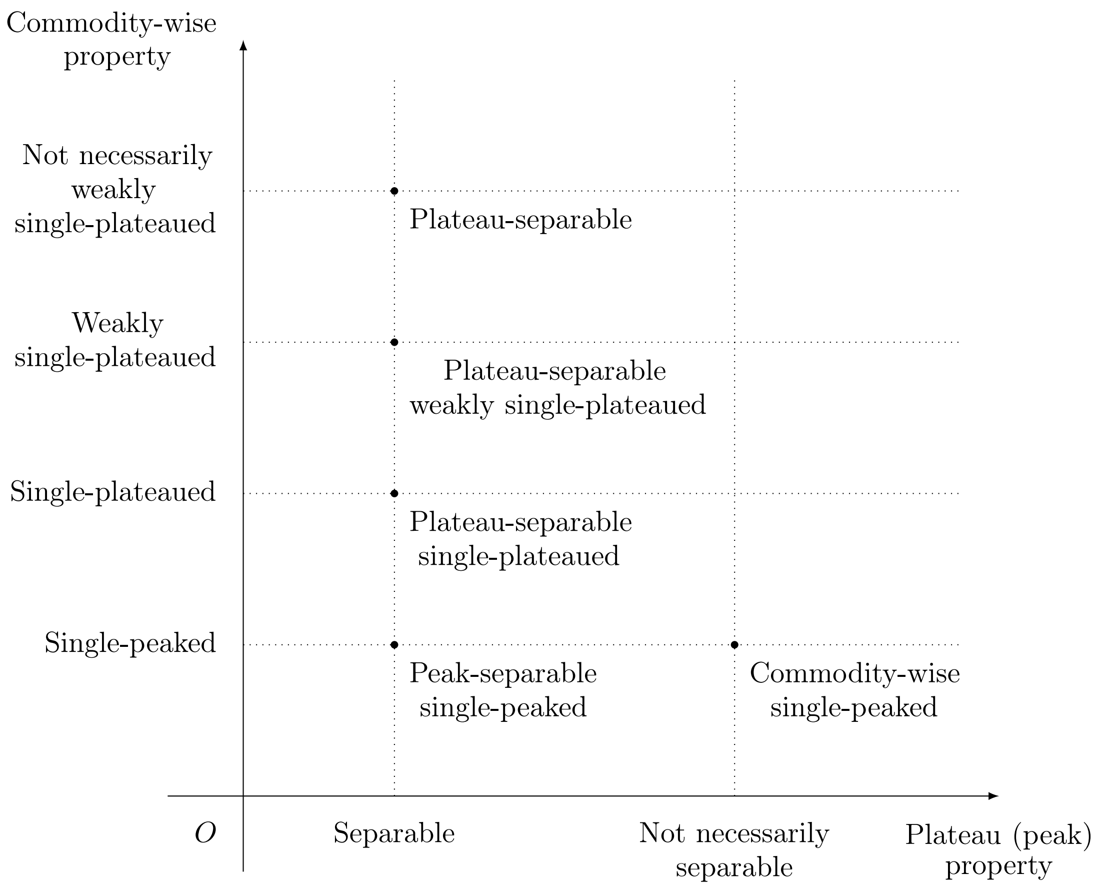


Figure 2: **Various classes of preferences.** The horizontal axis measures the plateau (peak) property of preferences, and the vertical axis the commodity-wise property. In the coordinate system so defined, the greater the abscissa or ordinate of preferences, the weaker the corresponding property.

Commodity-wise same-sidedness: For each $(R,\Omega) \in \mathcal{R}^N \times \mathbb{R}^L_+$ and each $\ell \in L$, one of the following holds: (i) for each $i \in N$, $\varphi_{i\ell}(R,\Omega) \leq \underline{p}_\ell(x_{i,-\ell}|R_i)$; (ii) for each $i \in N$, $\underline{p}_\ell(x_{i,-\ell}|R_i) \leq \varphi_{i\ell}(R,\Omega) \leq \bar{p}_\ell(x_{i,-\ell}|R_i)$; or (iii) for each $i \in N$, $\varphi_{i\ell}(R,\Omega) \geq \bar{p}_\ell(x_{i,-\ell}|R_i)$.

Next are fairness requirements. First, agents with the same preferences be assigned bundles that they find indifferent according to their common preferences.

Equal treatment of equals in welfare terms: For each $(R,\Omega) \in \mathcal{R}^N \times \mathbb{R}^L_+$ and each pair $i,j \in N$, if $R_i = R_j$, then $\varphi_i(R,\Omega)\, I_i\, \varphi_j(R,\Omega)$.

Alternatively, we could require that agents with the same preferences should be assigned the same bundles.

Equal treatment of equals in physical terms: For each $(R,\Omega) \in \mathcal{R}^N \times \mathbb{R}^L_+$ and each $i,j \in N$, if $R_i = R_j$, then $\varphi_i(R,\Omega) = \varphi_j(R,\Omega)$.

Our next fairness concept allows us to compare how agents with difference preferences are treated: no agent should prefer someone else's assignment to his own (Foley, 1967).

No-envy: For each $(R,\Omega) \in \mathcal{R}^N \times \mathbb{R}^L_+$ and each $i,j \in N$, $\varphi_i(R,\Omega)\, R_i\, \varphi_j(R,\Omega)$.

Clearly, each of *equal treatment of equals in physical terms* and *no-envy* implies *equal treatment of equals in welfare terms*, but the converse is not true. Also, *equal treatment of equals in physical terms* and *no-envy* are not logically related.

Finally, we consider a strategic requirement: no agent should ever be made better off by lying about her preferences. More precisely, for each preference profile, in the direct revelation game form associated with the rule, it is a weakly dominant strategy for each agent to report her true preferences.

Strategy-proofness: For each $(R,\Omega) \in \mathcal{R}^N \times \mathbb{R}^L_+$, each $i \in N$, and each $R'_i \in \mathcal{R}$, $\varphi_i(R,\Omega)\, R_i\, \varphi_i(R'_i, R_{-i}, \Omega)$.

3 Generalization of the Uniform Rule

3.1 Existence

First, we address the issue of existence of commodity-wise uniform allocations. On the domain of plateau-separable preferences, these allocation are obtained by simply applying the one-dimensional uniform rule commodity by commodity.

Commodity-wise uniform rule, U, on $\mathcal{R}^N_{pl.sep} \times \mathbb{R}^L_+$: For each $(R,\Omega) \in \mathcal{R}^N_{pl.sep} \times \mathbb{R}^L_+$, each

$\ell \in L$, and each $i \in N$,

$$U_{i\ell}(R,\Omega) = \begin{cases} \min\{\underline{p}_\ell(R_i), \lambda_\ell\} & \text{if } \Omega_\ell \leq \sum_{i\in N} \underline{p}_\ell(R_i); \\ \min\{\underline{p}_\ell(R_i) + \lambda_\ell, \bar{p}_\ell(R_i)\} & \text{if } \sum_{i\in N} \underline{p}_\ell(R_i) < \Omega_\ell \leq \sum_{i\in N} \bar{p}_\ell(R_i); \\ \max\{\lambda_\ell, \bar{p}_\ell(R_i)\} & \text{otherwise,} \end{cases}$$

where for each $\ell \in L$, λ_ℓ is chosen to satisfy $\sum_{i\in N} U_{i\ell}(R,\Omega) = \Omega_\ell$.

The commodity-wise uniform rule is proposed by Amorós (2002) for peak-separable single-peaked preferences. Our definition applies to a larger preference domain, namely, plateau-separable preferences. When restricted to plateau-separable weakly single-plateaued preferences, the commodity-wise uniform rule satisfies *commodity-wise same-sidedness*, *equal treatment of equals in physical (welfare) terms*, *no-envy*, and *strategy-proofness*, but it violates *efficiency*; we omit the simple proofs of these facts.

Next, to define the commodity-wise uniform solution for commodity-wise single-peaked preferences, we work with general one-dimensional rules and extend them to the multi-commodity case. To that end, we describe the one-dimensional model briefly. This model is a special case of the model introduced in Section 2, when $\bar{\ell} = 1$. We require preferences to be one-dimensional single-peaked, that is, to belong to $\mathcal{R}_{1.sg.pk}$.

For each $R_0 \in \mathcal{R}_{sg.pk}$, each $\ell \in L$, and each $a_{L\setminus\{\ell\}} \in \mathbb{R}^{L\setminus\{\ell\}}$, let $\boldsymbol{R_0|_{a_{L\setminus\{\ell\}}}}$ be the restriction of R_0 to the affine set $\{(a_\ell, a_{L\setminus\{\ell\}}) \in \mathbb{R}^L_+ \colon a_\ell \in \mathbb{R}^{\{\ell\}}_+\}$. Since $R_0 \in \mathcal{R}_{sg.pk}$, $R_0|_{a_{L\setminus\{\ell\}}}$ is one-dimensional single-peaked, and therefore, $p(R_0|_{a_{L\setminus\{\ell\}}})$ denotes the amount of commodity ℓ that maximizes $R_0|_{a_{L\setminus\{\ell\}}}$ on $\mathbb{R}^{\{\ell\}}_+$.

Let $\varphi\colon \mathcal{R}^N_{1.sg.pk} \times \mathbb{R}_+ \to \mathbb{R}^N_+$ be a one-dimensional rule. We now extend φ to the multi-commodity model on the domain of commodity-wise single-peaked preferences.

Commodity-wise extension $\boldsymbol{\varphi^{cw}}$ of $\boldsymbol{\varphi}$ to $\boldsymbol{\mathcal{R}^N_{sg.pk}} \times \mathbb{R}^{\boldsymbol{L}}_+$: For each $(R,\Omega) \in \mathcal{R}^N_{sg.pk} \times \mathbb{R}^L_+$ and each $x \in X$, $x \in \varphi^{cw}(R,\Omega)$ if for each $\ell \in L$, $(x_{i\ell})_{i\in N} = \varphi\Big((R_i|_{x_{iL\setminus\{\ell\}}})_{i\in N}, \Omega_\ell\Big)$.

Note that a commodity-wise extension is defined as a solution correspondence, because there is no guarantee that it is *single-valued.* With regard to non-empty-valuedness, Theorem 1 below provides a sufficient condition. As an application, the **commodity-wise uniform solution**, $\boldsymbol{U}$, is the commodity-wise extension of the one-dimensional uniform rule to $\mathcal{R}^N_{sg.pk} \times \mathbb{R}^L_+$. For each $(R,\Omega) \in \mathcal{R}^N_{sg.pk} \times \mathbb{R}^L_+$, an allocation $x \in U(R,\Omega)$ is called a **commodity-wise uniform allocation** for (R,Ω). On subdomains on which *single-valuedness* holds, we refer to this mapping as the **commodity-wise uniform rule**. The commodity-wise uniform rule or solution is defined on two domains, namely $\mathcal{R}^N_{pl.sep} \times \mathbb{R}^L_+$ and $\mathcal{R}^N_{sg.pk} \times \mathbb{R}^L_+$, and one can easily check that the two definitions coincide on their intersection.

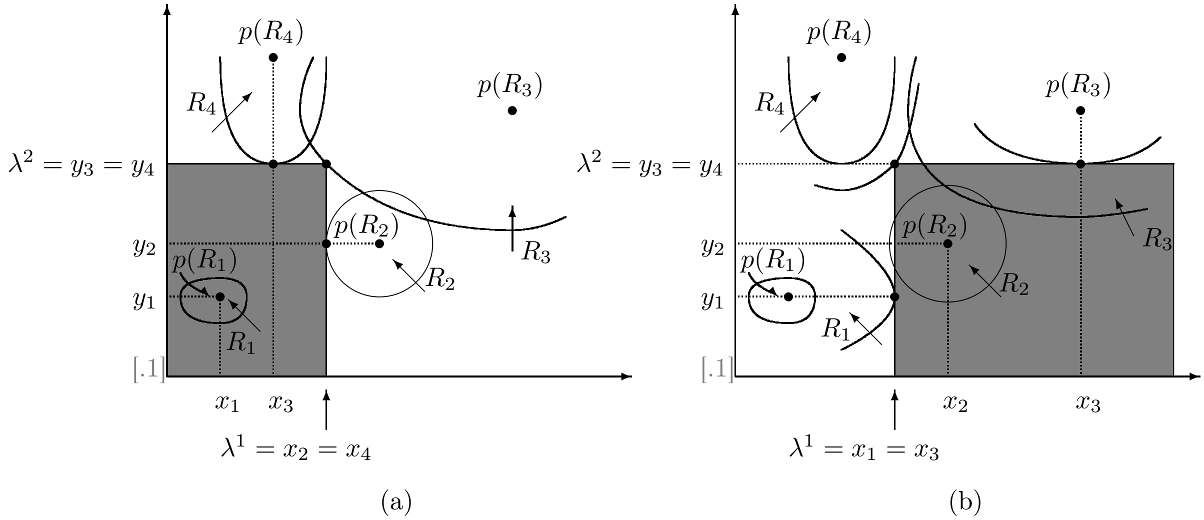


Figure 3: **Defining the commodity-wise uniform rule on the commodity-wise single-peaked and peak-separable domain.** (a) Here, there is not enough of either commodity. Then, upper bounds λ^1 and λ^2 in $\mathbb{R}_+$ are chosen and each agent maximizes his preferences in the rectangle $[0, \lambda^1] \times [0, \lambda^2]$. The bounds are specified so that the sum of the maximizers equals the social endowment. (b) Here, there is too much of commodity 1 but not enough of commodity 2. Then, a lower bound λ^1 is chosen for commodity 1 and an upper bound λ^2 is chosen for commodity 2: maximization takes place in the rectangle $[\lambda^1, \infty] \times [0, \lambda^2]$. Once again, the bounds are specified so that the sum of the maximizers equals the social endowment.

Now we identify a property of a one-dimensional rule (not necessarily the uniform rule) that ensures the non-empty-valuedness of its commodity-wise extension. It says that for each endowment, if a sequence of profiles of preferences is such that its associated sequence of peak amounts changes in a continuous manner, then so should the corresponding allocation.

Peak-continuity: For each $(R, \Omega) \in \mathcal{R}^N_{1.sg.pk} \times \mathbb{R}_+$ and each sequence of preference profiles $\{R^k\}_{k\in\mathbb{N}}$ in $\mathcal{R}^N_{1.sg.pk}$ such that for each $i \in N$, $\lim_{k\to\infty} p(R^k_i) = p(R_i)$, $\lim_{k\to\infty} \varphi(R^k, \Omega) = \varphi(R, \Omega)$.

Peak-continuity implies, in particular, "*peak-onliness*", the requirement that a rule determine an allocation based solely on the profile of peak amounts of preferences and the endowment. The following theorem states that if a one-dimensional rule is *peak-continuous*, then on the domain of commodity-wise single-peaked preferences, it has a well-defined commodity-wise extension. Because, for each commodity, an agent's peak amount depends on his consumption of the other commodities, proving existence requires invoking a fixed point argument.

Theorem 1. *Let $\varphi\colon \mathcal{R}^N_{1.sg.pk} \times \mathbb{R}_+$ be a* peak-continuous *one-dimensional rule. Then the commodity-wise extension φ^{cw} of φ to $\mathcal{R}^N_{sg.pk} \times \mathbb{R}^L_+$ is well-defined.*

Proof. Let $\varphi\colon \mathcal{R}^N_{1.sg.pk} \times \mathbb{R}_+$ be as in the theorem. Let $(R, \Omega) \in \mathcal{R}^N_{sg.pk} \times \mathbb{R}^L_+$ and $x \in X$. Let $\ell \in L$. For each $i \in N$ and each $\ell \in L$, treat $p(R_i|.)$ as a function which maps each vector $a_{L\setminus\{\ell\}} \in \mathbb{R}^{L\setminus\{\ell\}}_+$ to a point $p(R_i|_{a_{L\setminus\{\ell\}}}) \in \mathbb{R}_+$. Since R_i is continuous, it follows from the Berge maximum theorem that so is the function $p(R_i|.)$. Thus, the function that associates with x the

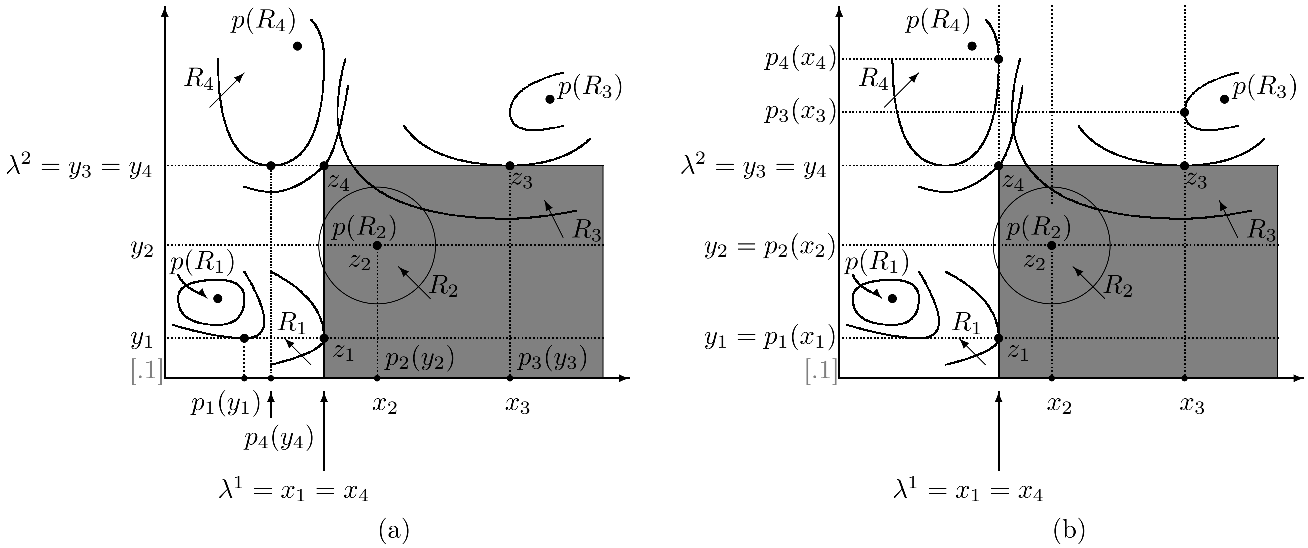


Figure 4: **Defining the commodity-wise uniform rule for an economy with commodity-wise single-peaked but not peak-separable preferences.** The allocation z is a commodity-wise uniform allocation for R. (a) Here, given the amounts consumed of commodity 2, there is not enough of commodity 1. For each agent, we identify his most preferred amount of commodity 1 given his consumption of commodity 2. (b) Here, given the amounts consumed of commodity 1, there is not enough of commodity 2.

vector $y_\ell \equiv \varphi(R_1|_{x_{1L\setminus\{\ell\}}}), \cdots, p(R_n|_{x_{nL\setminus\{\ell\}}}))$ is continuous. This is true for each $\ell \in L$. Thus, the function that associates with x the vector $y \equiv (y_\ell)_{\ell \in L}$ is a continuous function from X into itself. Further, the set X is a compact and convex subset of a Euclidean space. By the Brouwer fixed point theorem, the function has at least one fixed point. Each fixed point is an element of $\varphi^{cw}(R, \Omega)$ $\square$

Since the one-dimensional uniform rule is *peak-continuous*, a corollary to Theorem 1 is that uniform allocations exist on $\mathcal{R}^N_{sg.pk} \times \mathbb{R}^L_+$.

Corollary 1. *Uniform allocations exist on* $\mathcal{R}^N_{sg.pk} \times \mathbb{R}^L_+$.

Figure 3 shows a commodity-wise uniform allocation for an economy with peak-separable single-peaked preferences. It is obtained by simply applying the one-dimensional uniform rule commodity by commodity. On the other hand, when peak-separability fails, identifying a uniform allocation may not be so trivial. Figure 4 provides an example.

The literature on one-dimensional division problems offers a rich inventory of rules. Examples are the following. The "proportional rule" chooses the allocation at which assignments are proportional to peak amounts if at least one peak amount is positive and chooses equal division otherwise (preferences are the same then, so this choice is natural). The "constrained equal-distance rule" selects the efficient allocation at which the differences across agents between peak amount and consumption are equal unless an agent consumes nothing, in which case the difference for him could be smaller. The "constrained equal-preferred-sets rule" chooses the efficient allocation at which the sizes of the agents' upper contour sets at their assignments are equal, unless an agent

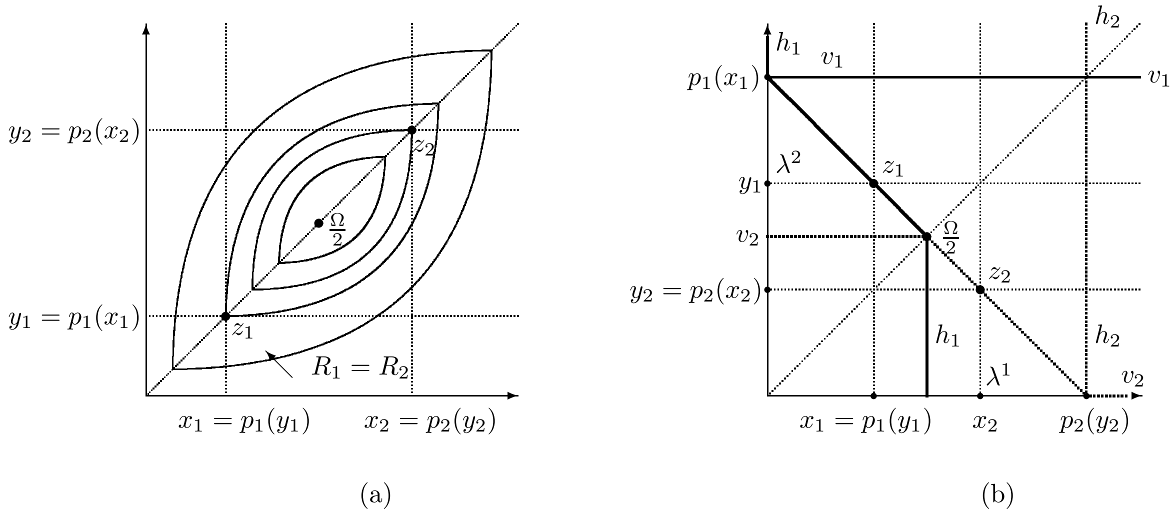


Figure 5: **Economies with multiple commodity-wise uniform allocations.** In both examples, there are a continuum of allocations satisfying the definition. (a) Here, loci $h - h$ and $v - v$ of each agent are confounded. At each of the commodity-wise uniform allocations, and in each dimension, both agents are satiated. (b) Here, loci $h - h$ and $v - v$ of each agent only meet at his global maximizer, so that his preferences can be specified to be smooth.

consumes nothing, in which case the size of his upper contour set could be smaller (Thomson, 1994). The proportional rule is not *peak-continuous* because of the special case when all peak amounts are zero, but a "symmetrized" variant of it, for which proportions are calculated from the origin if there is not enough of the commodity and from the social endowment if there is too much of the commodity (Thomson, 1994), is *peak-continuous*.

Because the constrained equal-distance and symmetrized proportional rules are *peak-continuous*, we can apply Theorem 1 to conclude that their commodity-wise extensions are well-defined solutions.

3.2 Uniqueness

Under peak separability, *single-valuedness* of commodity-wise extensions obviously holds. Without that assumption, however, uniqueness is not guaranteed. Two types of economies with multiple uniform allocations are shown in Figure 5. In panel (a), in each dimension, and for each amount of one of the commodities, each agent is satiated at a point on the 45° line. Preferences are not differentiable there. Panel (b) shows that multiplicity can occur with differentiable preferences. Indeed, the panel only shows the loci of kinks, but preferences can be specified so as to be differentiable.

The question we address here is whether weaker assumptions than peak separability can be identified under which uniqueness still holds. We provide a positive answer. By limiting the extent to which the amount of each commodity that maximizes an agent's preferences varies when his consumptions of the other commodities vary, a contraction mapping theorem becomes applicable, yielding uniqueness. It is natural to expect the absolute values of the slope of the loci of maximizers to matter. A formal argument confirming this conjecture is presented next.

A preference relation $R_0 \in \mathcal{R}_{sg.pk}$ is **Lipschitz peak-continuous** if there is $c \in \mathbb{R}_+$ such that for each $\ell \in L$ and each pair $x_{0L\setminus\{\ell\}}, y_{0L\setminus\{\ell\}} \in \mathbb{R}_+^{L\setminus\{\ell\}}$,

$$|p_\ell(x_{0L\setminus\{\ell\}}|R_0) - p_\ell(y_{0L\setminus\{\ell\}}|R_0)| \leq c||x_{0L\setminus\{\ell\}} - y_{0L\setminus\{\ell\}}||. \tag{1}$$

The infimum of the c's satisfying Equation (1) is the **Lipschitz constant for $\boldsymbol{R_0}$**. Let $\boldsymbol{\mathcal{R}_{sg.pk}(c)}$ denote the subclass of *Lipschitz peak-continuous* preferences in $\mathcal{R}_{sg.pk}$ whose Lipschitz constant is at most c.

Next is a requirement on one-dimensional rules. It says that a rule should respond to changes in the profile of peak amounts of preferences in a Lipschitz continuous manner. A one-dimensional rule $\varphi : \mathcal{R}_{1.sg.pk}^N \times \mathbb{R}_+ \to \mathbb{R}_+^N$ is **Lipschitz peak-continuous** if there is $c \in \mathbb{R}_+$ such that for each $R, R' \in \mathcal{R}_{1.sg.pk}^N$ and $\Omega \in \mathbb{R}_+$,

$$||\varphi(R,\Omega) - \varphi(R',\Omega)|| \leq c|| \left(p(R_i)\right)_{i\in N} - \left(p(R'_i)\right)_{i\in N} ||. \tag{2}$$

The infimum of the c's satisfying Equation ((2))is the **Lipschitz constant for $\boldsymbol{\varphi}$**. Note that if φ is *Lipschitz peak-continuous*, then it is *peak-continuous*. Also, if φ is *Lipschitz peak-continuous* with Lipschitz constant $c = 0$, then for each $\Omega \in \mathbb{R}_+$, $\varphi(\cdot,\Omega) : \mathcal{R}_{1.sg.pk}^N \to \mathbb{R}_+^N$ is a constant rule so that *single-valuedness* of the commodity-wise extension φ^{cw} of φ to $\mathcal{R}_{sg.pk}^N \times \mathbb{R}_+^L$ follows trivially. In what follows, we discuss only the case $c > 0$. Here is our uniqueness result:

Theorem 2. *Let $c \in \mathbb{R}_{++}$ and let $\varphi\colon \mathcal{R}_{1.sg.pk}^N \times \mathbb{R}_+ \to \mathbb{R}_+^N$ be a* Lipschitz peak-continuous *one-dimensional rule with Lipschitz constant c. Let $\mathcal{R}_* \equiv \bigcup_{c'} \mathcal{R}_{sg.pk}(c')$, where $c' \in [0, \frac{1}{c(\ell-1)}[$. Then the commodity-wise extension φ^{cw} of φ to $\mathcal{R}_{sg.pk}^N \times \mathbb{R}_+^L$ is* single-valued *on the subdomain $\mathcal{R}_*^N \times \mathbb{R}_+^L$.*

Proof. Let $c \in \mathbb{R}_{++}$, φ, and $\mathcal{R}_*$ be as in the theorem. Let $(R,\Omega) \in \mathcal{R}_*^N \times \mathbb{R}_+^L$. Then there is $c' \in [0, \frac{1}{c(\ell-1)}[$ such that $R \in \mathcal{R}_{sg.pk}(c')^N$. Define the mapping $T\colon \mathbb{R}_+^{LN} \to \mathbb{R}_+^{LN}$ as follows: for each $x \in \mathbb{R}_+^{LN}$, let $T(x) = y$, where for each $\ell \in L$, $y_\ell \equiv \varphi(R_1|_{x_{1L\setminus\{\ell\}}}, \cdots, R_n|_{x_{nL\setminus\{\ell\}}}, \Omega_\ell) \in \mathbb{R}_+^N$. Then $\varphi^{cw}(R,\Omega)$ is the set of fixed points of T. Thus, if we show that T is a contraction mapping, then by the Banach fixed point theorem, it has a unique fixed point and φ^{cw} is *single-valued*.

To show that T is a contraction, let $x, w \in \mathbb{R}_+^{LN}$, $y \equiv T(x)$ and $z \equiv T(w)$. Observe that

$$||T(x) - T(w)|| = \left[\sum_{i\in N}\sum_{\ell\in L}(y_{i\ell} - z_{i\ell})^2\right]^{\frac{1}{2}} = \left[\sum_{\ell\in L}\sum_{i\in N}(y_{i\ell} - z_{i\ell})^2\right]^{\frac{1}{2}} \tag{3}$$

and that for each $\ell \in L$,

$$\begin{aligned}
\sum_{i\in N}(y_{i\ell}-z_{i\ell})^2 &= \left\|\varphi\left(R_1|_{x_{1L\setminus\{\ell\}}},\cdots,R_n|_{x_{nL\setminus\{\ell\}}},\Omega_\ell\right)-\varphi\left(R_1|_{w_{1L\setminus\{\ell\}}},\cdots,R_n|_{w_{nL\setminus\{\ell\}}},\Omega_\ell\right)\right\|^2 \\
&\leq c^2\left\|\left(p(R_i|_{x_{iL\setminus\{\ell\}}})\right)_{i\in N}-\left(p(R_i|_{w_{iL\setminus\{\ell\}}})\right)_{i\in N}\right\|^2 \\
&= c^2\left\|\left(p_\ell(x_{iL\setminus\{\ell\}}|R_i)\right)_{i\in N}-\left(p_\ell(w_{iL\setminus\{\ell\}}|R_i)\right)_{i\in N}\right\|^2 \\
&= c^2\sum_{i\in N}\left[p_\ell(x_{iL\setminus\{\ell\}}|R_i)-(p_\ell(w_{iL\setminus\{\ell\}}|R_i)\right]^2 \\
&\leq c^2c'^2\sum_{i\in N}||x_{iL\setminus\{\ell\}}-w_{iL\setminus\{\ell\}}||^2,
\end{aligned}$$

where the first inequality holds because the Lipschitz constant for φ is c, and the last one because $R\in\mathcal{R}_{sg.pk}(c')^N$. Thus, equation (3) leads to

$$\begin{aligned}
||T(x)-T(w)|| &\leq cc'\left[\sum_{\ell\in L}\sum_{i\in N}||x_{iL\setminus\{\ell\}}-w_{iL\setminus\{\ell\}}||^2\right]^{\frac{1}{2}} \\
&= cc'(\bar{\ell}-1)||x-w||;
\end{aligned}$$

that is, T is a contraction with modulus $\beta\equiv cc'(\bar{\ell}-1)<1$. □

With Theorem 2 at hand, identifying a subdomain on which the commodity-wise extension of the one-dimensional uniform rule is single-valued amounts to finding the Lipschitz constant for the underlying one-dimensional rule. The next proposition shows that the Lipschitz constant for the one-dimensional uniform rule is at most $\sqrt{n}$.

Proposition 1. *The Lipschitz constant for the one-dimensional uniform rule is at most $\sqrt{n}$.*

Proof. Let $V:\mathcal{R}^N_{1.sg.pk}\times\mathbb{R}_+\to\mathbb{R}^N_+$ be the one-dimensional uniform rule.
Step 1: *Let $R,R'\in\mathcal{R}^N_{1.sg.pk}$ and $\Omega\in\mathbb{R}_+$. If there is $i\in N$ such that $R_i\neq R'_i$ and $R_{-i}=R'_{-i}$, then $||V(R)-V(R')||\leq\sqrt{n}|p(R_i)-p(R'_i)|$.*

Note that $|V_i(R)-V_i(R')|\leq|p(R_i)-p(R'_i)|$ and that for each $j\in N\backslash\{i\}$, $|V_j(R)-V_j(R')|\leq|p(R_i)-p(R'_i)|$. Thus, $||V(R)-V(R')||\leq\sqrt{n}|p(R_i)-p(R'_i)|$.

Step 2: *Let $R,R'\in\mathcal{R}^N_{1.sg.pk}$ and $\Omega\in\mathbb{R}_+$. Then $||V(R)-V(R')||\leq\sqrt{n}||(p(R_i))_{i\in N}-(p(R'_i))_{i\in N}||$.*

Let $R^0\equiv R$, and for each $k\in\{1,\cdots,n\}$, let $R^k\in\mathcal{R}^N_{1.sg.pk}$ be the profile obtained from R^{k-1} by replacing R^{k-1}_k by R'_k; i.e., $R^1\equiv(R'_1,R_2,\cdots,R_n)$, $R^2\equiv(R'_1,R'_2,R_3,\cdots,R_n)$, $R^3\equiv(R'_1,R'_2,R'_3,R_4,\cdots,R_n),\cdots,R^n\equiv R'$. Then by Step 1, for each $k\in\{1,\cdots,n\}$, $||V(R^{k-1})-$

$V(R^k)|| \leq \sqrt{n}|p(R_k) - p(R'_k)|$. Then

$$\begin{aligned} ||V(R) - V(R')|| &\leq \sum_{k\in\{1,\cdots,n\}} ||V(R^{k-1}) - V(R^k)|| \\ &\leq \sum_{k\in\{1,\cdots,n\}} \sqrt{n}|p(R_k) - p(R'_k)| \\ &= \sqrt{n}||(p(R_i))_{i\in N} - (p(R'_i))_{i\in N}||. \end{aligned}$$

□

Combining Theorem 2 and Proposition 1 gives a subdomain on which the commodity-wise uniform solution is *single-valued*.

Corollary 2. *Let* $\mathcal{R}_* \equiv \bigcup_{c'} \mathcal{R}_{sg.pk}(c')$, *where* $c' \in [0, \frac{1}{\sqrt{n}(\ell-1)}[$. *Then the commodity-wise uniform solution* $U : \mathcal{R}^N_{sg.pk} \times \mathbb{R}^L_+ \rightrightarrows \mathbb{R}^{LN}_+$ *is* single-valued *on the subdomain* $\mathcal{R}^N_* \times \mathbb{R}^L_+$.

Remark 1. The condition defining the subdomain identified in Corollary 2 is only sufficient for *single-valuedness*, and there are two reasons why it is not necessity. First, Theorem 2 is proved by appealing to the Banach fixed point theorem, which is a sufficient, but not necessary, condition for mappings to have a unique fixed point. Second, Proposition 1 provides an upper bound on the Lipschitz constant for the one-dimensional uniform rule, but the Lipschitz constant can indeed be smaller than the bound, in which case the subdomain in Corollary 2 may be further enlarged without compromising *single-valuedness*. With regard to the second reason, we conjecture that the Lipschitz constant for the one-dimensional uniform rule is smaller than $\sqrt{n}$ because the proof of Proposition 1 does not rely heavily on how the rule in particular behaves. In fact, if a *peak-only* one-dimensional rule φ satisfies the following solidarity property, then it is *Lipschitz peak-continuous* with Lipschitz constant at most $\sqrt{n}$: when agent i changes his preferences from R_i to R'_i, his assignment changes at most by the difference between the two peak amounts and so does the assignment for each other agent.[2] The constrained equal distance and symmetrized proportional rules satisfy this property. △

3.3 Normative and Strategic Properties

The next question is what properties the commodity-wise uniform solution, or rule when *single-valued*, inherits from the one-dimensional uniform rule. We know that *efficiency* is lost, although *commodity-wise same-sidedness* is preserved.

[2]Formally, this property can be stated as follows: for each $(R, \Omega) \in \mathcal{R}^N_{1.sg.pk} \times \mathbb{R}_+$, each $i \in N$, and each $R'_i \in \mathcal{R}_{1.sg.pk}$, (i) $|\varphi_i(R,\Omega) - \varphi_i(R'_i, R_{-i}, \Omega)| \leq |p(R_i) - p(R'_i)|$ and (ii) for each $j \in N\backslash\{i\}$, $|\varphi_j(R,\Omega) - \varphi_j(R'_i, R_{-i}, \Omega)| \leq |p(R_i) - p(R'_i)|$.

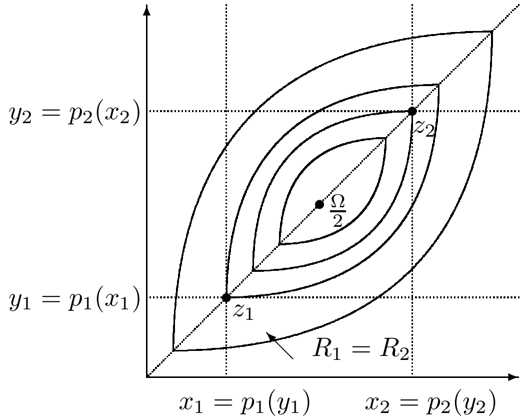


Figure 6: **Without peak-separability, a commodity-wise uniform allocation may not be envy-free and it may not meet the individual-endowments lower bounds.** The bundle z_1 has the property that keeping agent 1's consumption of commodity 2 fixed at y_1, agent 1 maximizes his preferences with respect to commodity 1 at x_1 and conversely. The same is true of the other agent. Yet, z_2 does not maximize R_2 over a product of intervals of the form $[\lambda^1, M_1] \times [0, \lambda^2]$. At z, agent 1 envies agent 2. Also, each agent would prefer the point of equal division to his assignment.

Whether or not uniform allocations are envy-free and whether they meet the *equal-division lower bound* depends on the range of permissible preferences. Figure 6 represents an economy in which both the *no-envy* and the *equal-division lower bound* properties are violated. Note that agent 2's indifference curve through his assignment has a kink at that point. Under smoothness, this would not occur. Alternatively, if the two loci meet only at the global maximizer, the double maximization defining the uniform allocation can occur at a point in the relative interior of a budget set only if this point is the global maximizer of the relation. For agents maximizing on a common budget set, *no-envy* is automatically met and if the budget set is convex, so is the *equal-division lower bound.*

Under peak separability, it is clear that *one-sided resource-monotonicity*, *one-sided population-monotonicity*, and *one-sided welfare-domination under preference-replacement* hold, because the one-dimensional uniform rule has these properties. *Consistency* and its *converse* hold whether or not peak separability is imposed.[3]

Concerning strategic issues, we have the following. Under peak separability, attainable sets are products of intervals and *strategy-proofness* holds (Amorós, 2002; Anno and Sasaki, 2009; Morimoto, Serizawa, and Ching, 2013). Without peak separability, this is not true anymore. In

[3]The first three properties are solidarity properties pertaining to changes in various parameters of the problem. *Resource-monotonicity* says that if the endowment varies, the welfare of all agents should be affected in the same direction. The other two are similar requirements pertaining to changes in population or in the preferences of some agents. The "one-sided" prefix limits the extent of the change. It says that the direction of the inequality between endowment and sum of peak amounts should not be reversed. *Consistency* says that the desirability of an allocation should not be affected by the departure of some agents with their assignments: in the "reduced economies" in which the amount to divide is what remains of the social endowment, each of the remaining agents should be assigned the same amount as he was initially. *Converse consistency* says that the desirability of an allocation for some economy can be deduced from the desirability of all of its restrictions to the two-agent reduced economies associated with it.

fact, *strategy-proofness* fails in that case. Moreover, any departure from peak separability leads to a violation of this property, as shown next.

4 Maximal Domain Results

Next, we define the notion of a "maximal domain of preferences for a list of properties". Let $\mathcal{R}_* \subset \hat{\mathcal{R}} \subset \mathcal{R}$. Domain $\mathcal{R}_*^N$ is a **maximal domain of preferences in $\hat{\mathcal{R}}^N$ for a list of axioms** if (i) there is a rule $\varphi\colon \mathcal{R}_*^N \times \mathbb{R}_+^L \to \mathbb{R}_+^{LN}$ satisfying the axioms; and (ii) for each $R_0 \in \hat{\mathcal{R}}\backslash\mathcal{R}_*$, no rule $\varphi\colon (\mathcal{R}_* \cup \{R_0\})^N \times \mathbb{R}_+^L \to \mathbb{R}_+^{LN}$ satisfies them. Our maximal domain results rely crucially on characterizations of the commodity-wise uniform rule, and therefore, we begin by reviewing them. These characterizations differ for $n = 2$ and $n > 2$.

Proposition 2. [Amorós, 2002] *Let $n \equiv 2$. Then the commodity-wise uniform rule is the only rule defined on $\mathcal{R}^N_{pk.sep-sg.pk} \times \mathbb{R}^L_+$ satisfying* commodity-wise-wise same-sidedness*,* equal treatment of equals in physical terms*, and* strategy-proofness*.*

For three or more agents, a characterization is available involving a more demanding fairness requirement, *no-envy* instead of *equal treatment of equals.*

Proposition 3. [Adachi, 2010] *The commodity-wise uniform rule is the only rule defined on $\mathcal{R}^N_{pk.sep-sg.pk} \times \mathbb{R}^L_+$ satisfying* commodity-wisewise same-sidedness*,* no-envy*, and* strategy-proofness*.*

The above two characterizations concern peak-separable single-peaked preferences. It is an open question whether, for three or more agents, *no-envy* can be weakened to *equal treatment of equals in physical terms* in characterizing the commodity-wise uniform rule.

Now to explore the existence of maximal domains of preferences, we take $\mathcal{R}_{pk.sep-sg.pk}$ as the base domain, and inquire about the extent to which conditions SP and C can be relaxed without compromising the compatibility of *commodity-wise same-sidedness*, *no-envy* (or *equal treatment of equals in physical terms*) and *strategy-proofness*. First, we examine relaxing condition C while maintaining SP. The following result, which involves the axioms in Amorós' (2002) characterization, shows that if $n \equiv 2$ and we add any preference relation $R_0 \in \mathcal{R}_{sg.pk}\backslash\mathcal{R}_{pk.sep-sg.pk}$ to $\mathcal{R}_{pk.sep-sg.pk}$, then no rule satisfies his axioms.

Theorem 3. *Let $n \equiv 2$. Then $\mathcal{R}^N_{pk.sep-sg.pk}$ is a maximal domain of preferences in $\mathcal{R}^N_{sg.pk}$ for* commodity-wise same-sidedness*,* equal treatment of equals in physical terms*, and* strategy-proofness*.*

Next, replacing *equal treatment of equals in physical terms* by the stronger fairness axiom, *no-envy*, we obtain a maximal domain result for the general n-agent case.

Theorem 4. *Let $n \geq 2$. Then $\mathcal{R}^N_{pk.sep-sg.pk}$ is a maximal domain of preferences in $\mathcal{R}^N_{sg.pk}$ for* commodity-wise-same-sidedness*,* no-envy*, and* strategy-proofness*.*

Theorems 3 and 4 show that in the presence of condition SP, condition C cannot be weakened if the axioms under consideration are to remain compatible.

Remark 2. Although Theorem 4 covers the two-agent case, it does not imply Theorem 3. This is because there is no logical relation between *equal treatment of equals in physical terms* and *no-envy*. Similar comments apply to Theorems 5 and 6 below. △

Remark 3. Theorems 3 and 4 show that $\mathcal{R}^N_{pk.sep-sg.pk}$ is a maximal domain of preferences in $\mathcal{R}^N_{sg.pk}$ for the respective axioms, but do not fully show the significance of the impossibility. The proof of the theorems in Section 5 may help answer the questions of this kind. When we impose the axioms of Theorems 3 and 4, the addition of any $R_0 \in \mathcal{R}_{sg.pk}$ to $\mathcal{R}_{pk.sep-sg.pk}$ creates an opportunity for other agents with separable quadratic preferences to profitably misreport, hence violating *strategy-proofness*. Put differently, the impossibility holds at such a fundamental level that when R_0 is added to the class of separable quadratic preferences, which is much smaller than $\mathcal{R}_{pk.sep-sg.pk}$, the axioms become incompatible. Nevertheless, since they are compatible on $\mathcal{R}^N_{pk.sep-sg.pk}$, the N-fold product of the class of quadratic preferences is not a maximal domain in $\mathcal{R}^N_{sg.pk}$. Similar comments apply to Theorems 5 and 6 below. △

Now we search for a maximal domain in the other direction, by relaxing condition SP, but not condition C. With C still in place, our inquiry is akin to the maximal domain questions in one-commodity models that have been answered by Ching and Serizawa (1998), and Massó and Neme (2001, 2004). Ching and Serizawa (1998)'s study is closest in spirit to ours, because we allow the endowment to vary, as they do. They show that in the one-commodity case, the domain of single-plateaued preferences is a maximal subdomain of the domain of continuous preferences for *efficiency*, *equal treatment of equals in welfare terms*, and *strategy-proofness*. For two or more commodities, substituting *commodity-wise same-sidedness* for *efficiency* further enlarges the maximal domain, to include plateau-separable weakly single-plateaued preferences. Also, depending on the number of agents, different fairness axioms are imposed. Theorem 5 pertains to the two-agent case.

Theorem 5. *Let $n \equiv 2$. Then $\mathcal{R}^N_{pl.sep-w.sg.pl}$ is a maximal domain of preferences in $\mathcal{R}^N_{pl.sep}$ for* commodity-wise same-sidedness*,* equal treatment of equals in physical terms*, and* strategy-proofness*.*

We omit the proof of Theorem 5 because with only two agents, a simpler version of the proof of Theorem 6 suffices. For its proof, we invoke Proposition 2 instead of Proposition 3. As was the case when we only weakened condition C, our result for the general n-agent case involves replacing *equal treatment of equals in physical terms* by *no-envy.*

Theorem 6. *Let $n \geq 2$. Then $\mathcal{R}^N_{pl.sep-w.sg.pl}$ is a maximal domain of preferences in $\mathcal{R}^N_{pl.sep}$ for* commodity-wise same-sidedness*,* no-envy*, and* strategy-proofness.

Remark 4. If we impose *efficiency* instead of *commodity-wise same-sidedness*, Theorems 3-6 turn into impossibilities. Indeed, when $n \equiv 2$, *efficiency* and *strategy-proofness* imply dictatorship on an even smaller preference domain, namely $\mathcal{R}^N_{pk.sep-sg.pk}$ (Amorós, 2002). When $n \geq 2$, on the domain of continuous, monotonic, convex, homothetic and smooth preferences, *efficiency* and *strategy-proofness* imply a violation of the requirement that each agent be provided a "minimal consumption guarantee", a requirement that is implied by *no-envy* (Serizawa and Weymark, 2003). The proof of this result extends to $\mathcal{R}^N_{pk.sep-sg.pk}$, yielding an impossibility on this domain. $\triangle$

Remark 5. In our maximal domain results, Theorems 3-6, the social endowment is allowed to vary. Alternatively, we could seek to identify maximal domains for each fixed endowment, as done by Massó and Neme (2001, 2004). We conjecture that something similar to the connection between Ching and Serizawa (1998) and Massó and Neme (2001) holds in the multi-commodity case as well. That is, given an endowment vector Ω, the maximal domain of preferences for the respective axioms is larger than those identified in Theorems 3-6. In particular, it permits some measure of freedom "sufficiently far" from $\frac{\Omega}{n}$. of course, the intersection of those maximal domains obtained by letting Ω vary over $\mathbb{R}^L_+$ gives the domains our theorems identify. $\triangle$

5 Proofs

We use the following concepts and notation. Given a preference domain $\mathcal{R}$, a rule $\varphi\colon \mathcal{R}^N \times \mathbb{R}^L_+ \to \mathbb{R}^{LN}_+$, and an endowment vector $\Omega \in \mathbb{R}^L_+$, for each $i \in N$ and each $R_{-i} \in \mathcal{R}^{N\setminus\{i\}}$, let $\boldsymbol{A_i(R_{-i}, \Omega, \varphi)} \equiv \{\varphi_i(R_i, R_{-i}, \Omega)\colon R_i \in \mathcal{R}\}$ be **agent $\boldsymbol{i}$'s attainable set given $\boldsymbol{(R_{-i}, \Omega)}$ under $\boldsymbol{\varphi}$**. Also, given $a \in \mathbb{R}^L_+$ and $\Omega \in \mathbb{R}^L_+$, let $\boldsymbol{\sigma(a)} \equiv \Omega - a$ be the symmetric image of a with respect to $\frac{\Omega}{2}$. Given $a, b \in \mathbb{R}^L_+$, let $\mathbf{seg}[\boldsymbol{a}, \boldsymbol{b}] \equiv \{ta + (1-t)b : t \in [0,1]\}$ be the line segment connecting a and b; let $\mathbf{box}[\boldsymbol{a}, \boldsymbol{b}]$ be the smallest box containing a and b with sides parallel to the axes.

Also, to mean for instance that agent 1, with true preferences R_1, and facing agents $2, \ldots, n$ announcing $R_2, \ldots, R_n$, respectively, is better off announcing R'_1 than telling the truth, we write

$$\varphi_1(\overset{\text{lie}}{R'_1}, R_2, \ldots, R_n) \overset{\text{truth}}{P_1} \varphi_1(\overset{\text{truth}}{R_1}, R_2, \ldots, R_n).$$

We only prove the result for $\bar{\ell} = 2$; adapting the argument for $\bar{\ell} > 2$ is simple. Thus, throughout this section, let $L \equiv \{1, 2\}$.

5.1 Proof of Theorem 3

Let $N \equiv \{1, 2\}$. Let $R_0 \in \mathcal{R}_{sg.pk} \backslash \mathcal{R}_{pk.sep-sg.pk}$, and suppose, by contradiction, that there is a rule φ defined on $(\mathcal{R}_{pk.sep-sg.pk} \cup \{R_0\})^N \times \mathbb{R}^L_+$ that satisfies the axioms listed in the theorem. Note that either $v(\cdot|R_0)$ or $h(\cdot|R_0)$ is not constant. Without loss of generality, assume that $v(\cdot|R_0)$ is not. Let $R_1 \equiv R_0$. We distinguish four cases.

Case 1: (Figure 7) *There is $\hat{t} > p_1(R_1)$ such that $v(\hat{t}|R_1) < p_2(R_1)$.*

Let $\Omega \equiv \big(2\hat{t}, 2p_2(R_1)\big) \in \mathbb{R}^L_+$. We proceed in three steps.

Step 1: *Let $R_2 \in \mathcal{R}_{pk.sep-sg.pk}$ be such that $p_1(R_2) = \hat{t}$ and $p_2(R_2) > \Omega_2$, and let $x \equiv \varphi(R_1, R_2, \Omega)$. Then $x_1 = \big(\hat{t}, v(\hat{t}|R_1)\big)$ and $x_2 = \Omega - x_1$.*

To see this, note first that when facing R_2, if agent 1 announces a preference relation in $\mathcal{R}_{pk.sep-sg.pk}$, then by Proposition 2, $\varphi(R_1, R_2) = U(R_1, R_2)$. Thus, $A_1(R_2, \Omega, \varphi) \supseteq \{U_1(\tilde{R}_1, R_2, \Omega) : \tilde{R}_1 \in \mathcal{R}_{pk.sep-sg.pk}\} = \text{seg}[(\hat{t}, 0), \frac{\Omega}{2}]$. Within that segment, $a \equiv \big(\hat{t}, v(\hat{t}|R_1)\big)$ uniquely maximizes R_1. Thus, by *strategy-proofness* applied to agent 1 with true preferences R_1 and facing the announcement R_2, it follows that $x_1 \, R_1 \, a$.

To show that in fact, $x_1 = a$, suppose not. Because the R_1-indifference curve through a lies to the left of the vertical line through a, we have $x_{11} < a_1$. Let $\tilde{R}_1 \in \mathcal{R}_{pk.sep-sg.pk}$ be such that $p(\tilde{R}_1) = x_1$, and let $\tilde{x} \equiv \varphi(\tilde{R}_1, R_2, \Omega)$. Since $\frac{1}{2}\Big(p_1(\tilde{R}_1) + p_1(R_2)\Big) < \frac{\Omega_1}{2} = p_1(R_2)$, it follows that $\tilde{x}_{11} = \frac{\Omega_1}{2}$, so that $\tilde{x}_1 \neq p(\tilde{R}_1)$. Thus, $\varphi_1(\overset{\text{lie}}{R_1}, R_2) \overset{\text{truth}}{\tilde{P}_1} \varphi_1(\overset{\text{truth}}{\tilde{R}_1}, R_2)$, in violation of *strategy-proofness*.

Step 2: *Let $R_2' \in \mathcal{R}_{pk.sep-sg.pk}$ be such that $p(R_2') > \Omega$ and $x_2 \, P_2' \, (\Omega - p(R_1))$, and $x' \equiv \varphi(R_1, R_2', \Omega)$. Then $x_1' = p(R_1)$ and $x_2' = \Omega - x_1'$.*

By an argument similar to that in Step 1, $A_1(R_2', \varphi) \supseteq \text{box}[0, \frac{\Omega}{2}]$. Within that box, $p(R_1)$ uniquely maximizes R_1. Thus, by *strategy-proofness* applied to agent 1 with true preferences R_1 and facing the announcement R_2', it follows that $x_1' \, R_1 \, p(R_1)$. Hence, $x_1' = p(R_1)$.

Step 3: *Concluding.* By Steps 1 and 2, $\varphi_2(R_1, \overset{\text{lie}}{R_2}) \overset{\text{truth}}{P_2'} \varphi_2(R_1, \overset{\text{truth}}{R_2'})$, in violation of *strategy-proofness.*

Case 2: *There is $\hat{t} > p_1(R_1)$ such that $v(\hat{t}|R_1) > p_2(R_1)$.*
Case 3: *There is $\hat{t} < p_1(R_1)$ such that $v(\hat{t}|R_1) < p_2(R_1)$.*
Case 4: *There is $\hat{t} < p_1(R_1)$ such that $v(\hat{t}|R_1) > p_2(R_1)$.*

For each of Cases 2-4, an argument similar to that in Case 1 leads to a contradiction. We omit the details.

(a)

(b)

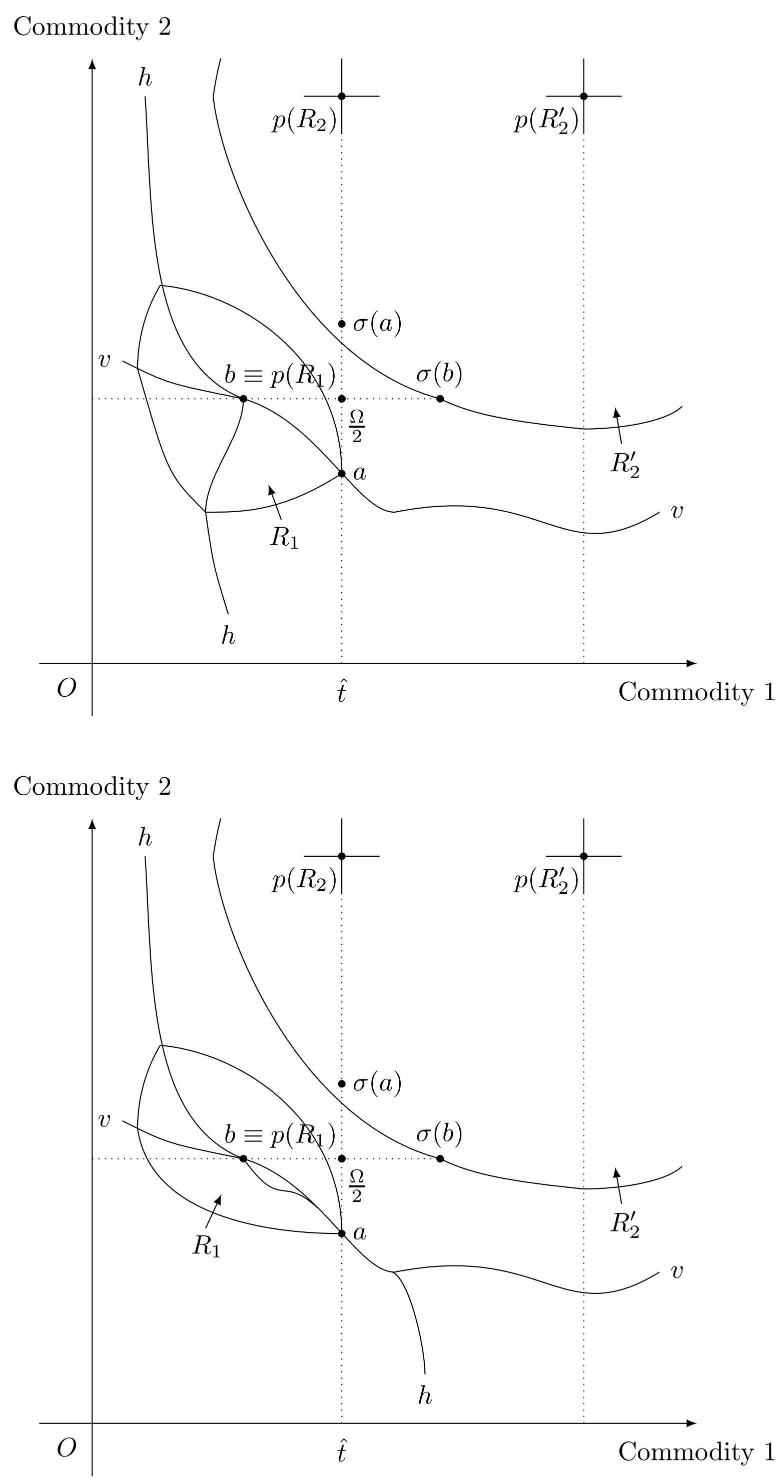


Figure 7: **Illustrating Case 1 in the proof of Theorem 3.** Panels (a) and (b) show two possible configurations of the R_1-indifference curve through a. Steps 1 and 2 show that $\varphi_2(R_1, R_2, \Omega) = \sigma(a)$ and $\varphi_2(R_1, R_2', \Omega) = \sigma(b)$. Since $\sigma(a)\, P_2'\, \sigma(b)$, *strategy-proofness* is violated.

5.2 Proof of Theorem 4

Let $R_0 \in \mathcal{R}_{sg.pk} \backslash \mathcal{R}_{pk.sep-sg.pk}$, and suppose, on the contrary, that there is a rule φ defined on $(\mathcal{R}_{pk.sep-sg.pk} \cup \{R_0\})^N \times \mathbb{R}^L_+$ that satisfies the axioms listed in the theorem. Note that either $v(\cdot|R_0)$ or $h(\cdot|R_0)$ is not constant. Without loss of generality, assume that $v(\cdot|R_0)$ is not. Let $R_1 \equiv R_0$. We distinguish four cases.

Case 1: (Figure 8) *There is* $\hat{t} > p_1(R_1)$ *such that* $v(\hat{t}|R_1) < p_2(R_1)$.

Let $\Omega \equiv \big(n \cdot \hat{t}, n \cdot p_2(R_1)\big) \in \mathbb{R}^L_+$. Let $a \equiv \big(\hat{t}, v(\hat{t}|R_1)\big)$, $b \equiv \frac{\Omega - a}{n-1}$, and $c \in \mathbb{R}^L_+$ be such that $c_1 \equiv 2\hat{t} - p_1(R_1)$, and $\frac{\Omega_2}{n} < c_2 < b_2$. Let $R'_0 \in \mathcal{R}_{pk.sep-sg.pk}$ be such that $p(R'_0) = (\hat{t}, \Omega_2)$ and $\frac{\Omega}{n}\, I'_0\, c$; let $R''_0 \in \mathcal{R}_{pk.sep-sg.pk}$ be such that $p(R''_0) = (c_1, \Omega_2)$ and $b\, R''_0\, c$. Now, for each $i \in N \backslash \{1, 2\}$, let $R_i \equiv R'_0$. We proceed in three steps.

Step 1: *Let* $R_2 \equiv R'_0$ *and* $x \equiv \varphi(R_1, R_2, \cdots, R_n, \Omega)$. *Then* $x_1 = a$, *and for each* $i \in N \backslash \{1\}$, $x_i = b$.

To see this, note first that when facing R_{-1}, if agent 1 announces a preference relation in $\mathcal{R}_{pk.sep-sg.pk}$, then by Proposition 3, φ allocates Ω according to the uniform rule. Thus, $A_1(R_{-1}, \Omega, \varphi) \supseteq \{U_1(\tilde{R}_1, R_{-1}, \Omega) \colon \tilde{R}_1 \in \mathcal{R}_{pk.sep-sg.pk}\} = \mathrm{seg}[(\hat{t}, 0), \frac{\Omega}{n}]$. Within that segment, a uniquely maximizes R_1. Thus, by *strategy-proofness* applied to agent 1 with true preferences R_1 and facing the announcements R_{-1}, it follows that $x_1\, R_1\, a$.

To show that in fact, $x_1 = a$, suppose not. Because the R_1-indifference curve through a lies to the left of the vertical through a, we have $x_{11} < a_1$. Let $\tilde{R}_1 \in \mathcal{R}_{pk.sep-sg.pk}$ be such that $p(\tilde{R}_1) = x_1$, and let $\tilde{x} \equiv \varphi(\tilde{R}_1, R_{-1}, \Omega)$. Since $p_1(\tilde{R}_1) < \frac{\Omega_1}{n} = p_1(R_2) = \cdots = p_1(R_n)$, it follows that $\tilde{x}_{11} = \frac{\Omega_1}{n}$ so that $\tilde{x}_1 \neq p(\tilde{R}_1)$. Thus,

$$\varphi_1(\overset{\text{lie}}{R_1}, R_2, \ldots, R_n) \overset{\text{truth}}{\tilde{P}_1} \varphi_1(\overset{\text{truth}}{\tilde{R}_1}, R_2, \ldots, R_n),$$

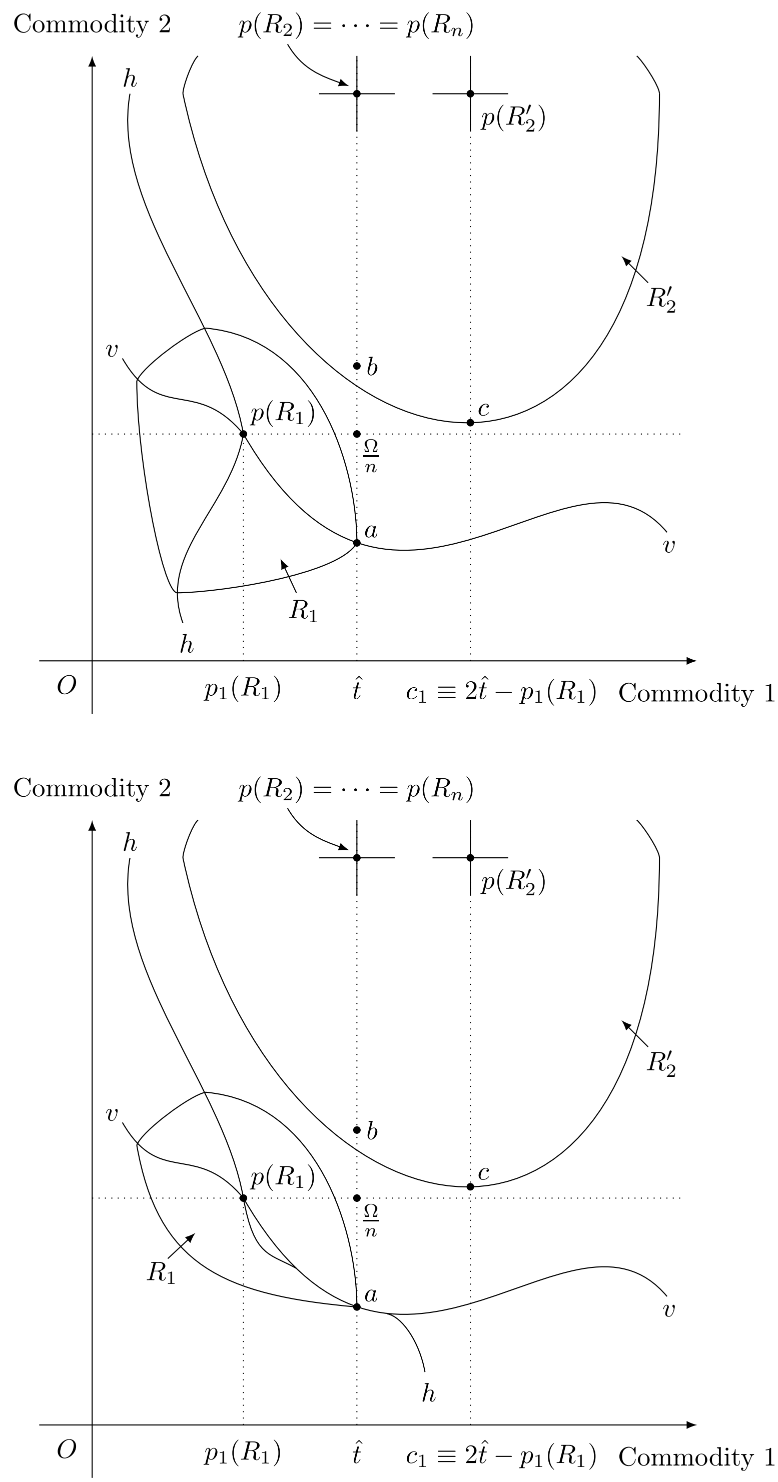
in violation of *strategy-proofness*.

Next, we show that for each $i \in N \backslash \{1\}$, $x_i = b$. If there is $i \in N \backslash \{1\}$ such that $x_{i1} < \frac{\Omega_1}{n}$, then by *commodity-wise same-sidedness*, for each $j \in N \backslash \{1, i\}$, $x_{j1} \leq \frac{\Omega_1}{n}$, so that $\sum_N x_{j1} < \frac{\Omega_1}{n} + (n-1)\frac{\Omega_1}{n} = \Omega_1$, violating feasibility. Thus, for each $i \in N \backslash \{1\}$, $x_{i1} \geq \frac{\Omega_1}{n}$, and a similar argument shows that the latter inequality is indeed an equality. Now by *no-envy* applied to agents $2, \cdots, n$ and feasibility, it follows that for each $i \in N \backslash \{1\}$, $x_i = b$.

Step 2: *Let* $R'_2 \equiv R''_0$, *and* $x' \equiv \varphi(R_1, R'_2, R_3, \cdots, R_n, \Omega)$. *Then* $x'_1 = p(R_1)$ *and* $x'_2 \in \mathrm{seg}\,[(c_1, 0), c]$.

Assume first that $n \geq 3$. By an argument similar to that in Step 1, $A_1(R'_2, R_3, \cdots, R_n, \Omega, \varphi) \supseteq \mathrm{box}\,\big[(p_1(R_1), 0), \frac{\Omega}{n}\big]$. Within that box, $p(R_1)$ uniquely maximizes R_1. Thus, by *strategy-proofness*

(a)



(b)

Figure 8: **Illustrating Case 1 in the proof of Theorem 4.** Panels (a) and (b) show two possible configurations of the R_1-indifference curve through bundle a. Steps 1 and 2 show that $\varphi_2(R_1, R_2, \cdots, R_n, \Omega) = b$ and $\varphi_2(R_1, R'_2, R_3, \cdots, R_n, \Omega) \in$ seg $[(c_1, 0), c]$. Since $b\, P'_2\, c$, *strategy-proofness* is violated.

applied to agent 1 with true preferences R_1 and facing the announcements $R'_2, R_3, \cdots, R_n$, it follows that $x'_1 \, R_1 \, p(R_1)$. Hence, $x'_1 = p(R_1)$.

To see $x'_2 \in \text{seg}\,[(c_1, 0), c]$, we first show that $x'_{21} = c_1$ and for each $i \in N \backslash \{1, 2\}$, $x'_{i1} = \frac{\Omega_1}{n}$. If $x'_{21} < c_1$, then by *commodity-wise same-sidedness*, for each $i \in N \backslash \{1, 2\}$, $x'_{i1} \leq \frac{\Omega_1}{n}$, so that $\sum_N x'_{j1} < p_1(R_1) + c_1 + (n-2)\frac{\Omega_1}{n} = \Omega_1$, violating feasibility. Thus, $x'_{21} \geq c_1$, and similar arguments show that the latter inequality is indeed an equality and that for each $i \in N \backslash \{1, 2\}$, $x'_{i1} = \frac{\Omega_1}{n}$.

Now, to show $x'_2 \in \text{seg}\,[(c_1, 0), c]$, suppose, on the contrary, that $x'_2 \in \text{seg}\,]c, (c_1, \Omega_2)]$. Because for each $i \in N \backslash \{1, 2\}$, $x'_{i1} = \frac{\Omega_1}{n}$, applying *no-envy* to agents $3, \cdots, n$, it follows that $x'_3 = \cdots = x'_n$. By feasibility, this common bundle lies in $\text{seg}\,\big[(\hat{t}, 0), \frac{\Omega}{n}\big[$, and agents $3, \cdots, n$ envy agent 2, in violation of *no-envy*. Thus, $x'_2 \in \text{seg}\,[(c_1, 0), c]$.

If $n = 2$, the above argument simplifies, and we obtain that $x'_1 = p(R_1)$ and $x'_2 = \Omega - x'_1 \in \text{seg}\,[(c_1, 0), c]$.

Step 3: *Concluding.* By Steps 1 and 2,

$$\varphi_2(R_1, \overset{\text{lie}}{R_2}, R_3, \ldots, R_n) \overset{\text{truth}}{P'_2} \varphi_2(R_1, \overset{\text{truth}}{R'_2}, R_3, \ldots, R_n),$$

in violation of *strategy-proofness*.

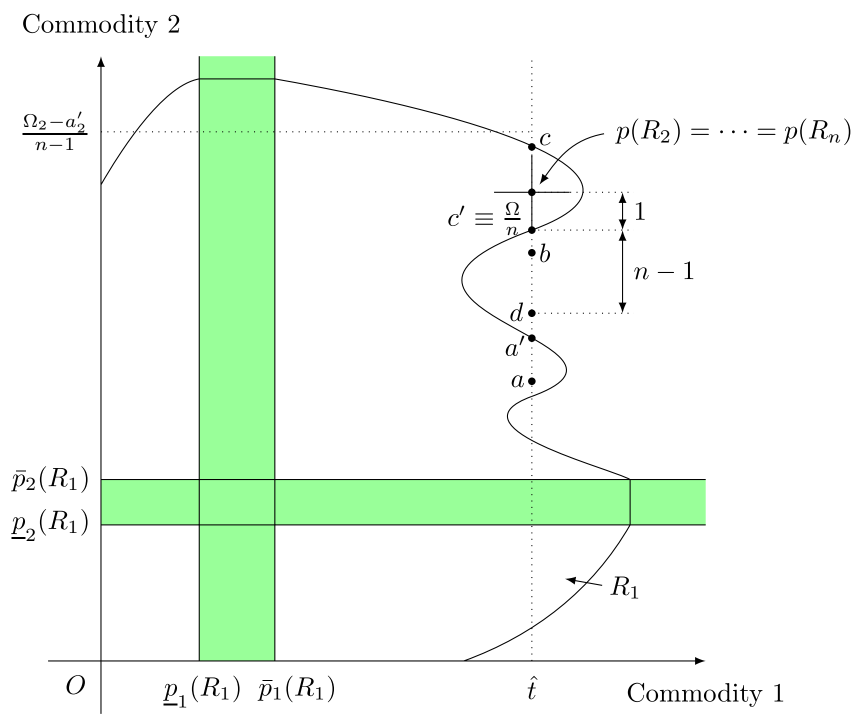
Case 2: *There is $\hat{t} > p_1(R_1)$ such that $v(\hat{t}|R_1) > p_2(R_1)$.*
Case 3: *There is $\hat{t} < p_1(R_1)$ such that $v(\hat{t}|R_1) < p_2(R_1)$.*
Case 4: *There is $\hat{t} < p_1(R_1)$ such that $v(\hat{t}|R_1) > p_2(R_1)$.*

For each of Cases 2-4, an argument similar to that in Case 1 leads to a contradiction. We omit the details.

5.3 Proof of Theorem 6

Let $R_0 \in \mathcal{R}_{pl.sep} \backslash \mathcal{R}_{pl.sep-w.sg.pl}$, and suppose, on the contrary, that there is a rule φ defined on $(\mathcal{R}_{pl.sep-w.sg.pl} \cup \{R_0\})^N \times \mathbb{R}^L_+$ that satisfies the axioms listed in the theorem. Let $R_1 \equiv R_0$, and note that there is $\hat{t} \in \mathbb{R}_+$ such that the restriction of R_0 to either $\{(\hat{t}, s) : s \in \mathbb{R}^{\{2\}}_+\}$ or $\{(s, \hat{t}) : s \in \mathbb{R}^{\{1\}}_+\}$ is not weakly single-plateaued. Assuming, without loss of generality, that the former is the case, there are $a, b, c \in \mathbb{R}^L_+$ such that $a_1 = b_1 = c_1 = \hat{t}$, $a_2 < b_2 < c_2$, $a \, P_1 \, b$, and $c \, P_1 \, b$. Define $a'_2 \equiv \max\{s \in [a_2, b_2] : (\hat{t}, s) \, I_1 \, a\}$ if $c \, R_1 \, a$, and $\max\{s \in [a_2, b_2] : (\hat{t}, s) \, I_1 \, c\}$ otherwise; $c'_2 \equiv \min\{s \in [b_2, c_2] : (\hat{t}, s) \, I_1 \, a\}$ if $c \, R_1 \, a$, and $\min\{s \in [b_2, c_2] : (\hat{t}, s) \, I_1 \, c\}$ otherwise. By continuity of R_1, a'_2 and c'_2 are well defined. Let $a' \equiv (\hat{t}, a'_2)$ and $c' \equiv (\hat{t}, c'_2)$. Because the restriction of R_1 to $\{(\hat{t}, s) : s \in \mathbb{R}^{\{2\}}_+\}$ is maximized on $[\underline{p}_2(R_1), \bar{p}_2(R_1)]$, it follows that either $a'_2 \geq \bar{p}_2(R_1)$ or $c'_2 \leq \underline{p}_2(R_1)$. We discuss these cases separately.

(a)



(b)

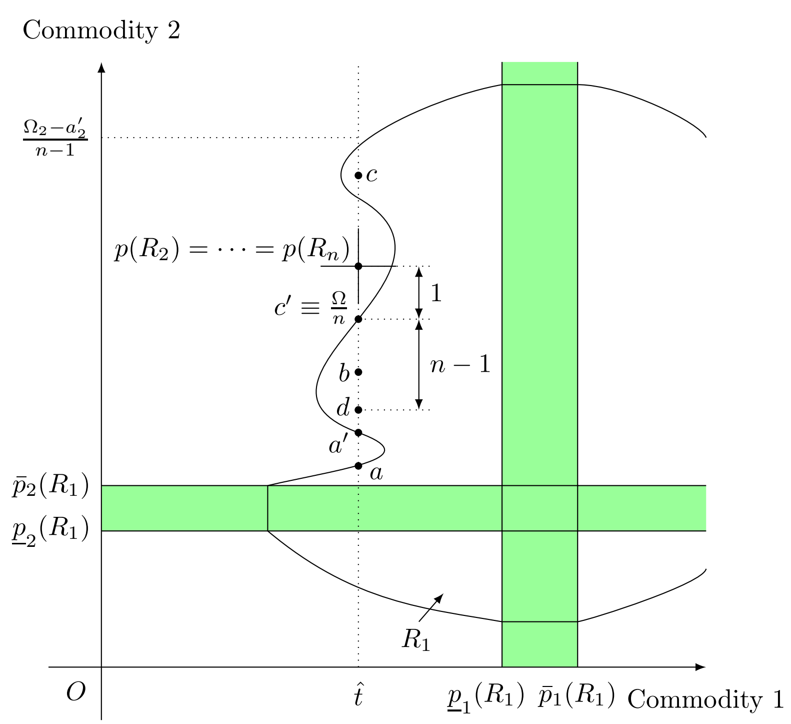


Figure 9: **Illustrating Case 1 in the proof of Theorem 6.** Panels (a) and (b) show two possible configurations of the R_1-indifference curve through bundles a' and c'. In either case, for each $i \in N$, $\varphi_i(R_1, \cdots, R_n, \Omega) = \frac{\Omega}{n}$, so that *commodity-wise same-sidedness* is violated.

Case 1: (Figure 9) $a'_2 \geq \bar{p}_2(R_1)$.

Let $\Omega \equiv nc' \in \mathbb{R}^L_+$. We first pick a preference relation in $\mathcal{R}_{pk.sep-sg.pk}$: let $R'_0 \in \mathcal{R}_{pk.sep-sg.pk}$ be such that $p_1(R'_0) = \hat{t}$ and $p_2(R'_0) \in]c'_2, \frac{\Omega_2 - a'_2}{n-1}[$. For each $i \in N\backslash\{1\}$, let $R_i \equiv R'_0$, and $x \equiv \varphi(R_1, R_2, \cdots, R_n)$. Below we show that for each $i \in N$, $x_i = \frac{\Omega}{n}$, which contradicts *commodity-wise same-sidedness*.

To prove the claim, first let $d \equiv \Omega - (n-1)p(R'_0)$. Note that when facing R_{-1}, if agent 1 announces a preference relation in $\mathcal{R}_{pk.sep-sg.pk}$, then by Proposition 3, φ allocates Ω according to the uniform rule. Thus, $A_1(R_{-1}, \Omega, \varphi) \supseteq \{U_1(\tilde{R}_1, R_{-1}, \Omega) : \tilde{R}_1 \in \mathcal{R}_{pk.sep-sg.pk}\} = \text{seg}[d, \frac{\Omega}{n}]$. Within that segment, $\frac{\Omega}{n}$ uniquely maximizes R_1. Thus, by *strategy-proofness* applied to agent 1 with true preferences and facing the announcements R_{-1}, it follows that $x_1 \; R_1 \; \frac{\Omega}{n}$.

To show that in fact, $x_1 = \frac{\Omega}{n}$, suppose not. Let $\tilde{R}_1 \in \mathcal{R}_{pk.sep-sg.pk}$ be such that $p(\tilde{R}_1) = x_1$, and let $\tilde{x} \equiv \varphi(\tilde{R}_1, R_{-1})$. If $x_{11} < \frac{\Omega_1}{n}$, then since $p_1(\tilde{R}_1) < \frac{\Omega_1}{n} = p_1(R_2) = \cdots = p_1(R_n)$, it follows that $\tilde{x}_{11} = \frac{\Omega_1}{n}$ so that $\tilde{x}_1 \neq p(\tilde{R}_1)$. Thus,

$$\varphi_1(\overset{\text{lie}}{R_1}, R_2, \ldots, R_n) \overset{\text{truth}}{\tilde{P}_1} \varphi_1(\overset{\text{truth}}{\tilde{R}_1}, R_2, \ldots, R_n),$$

in violation of *strategy-proofness*.

The cases $x_{11} > \frac{\Omega_1}{n}$ and $x_{12} > \frac{\Omega_2}{n}$ can be shown to yield a contradiction similarly. If $x_{12} < \frac{\Omega_2}{n}$, then because $x_{11} = \frac{\Omega_1}{n}$ and $x_1 \; R_1 \; \frac{\Omega}{n}$, we have $x_{12} \leq a'_2 < d_2$. Since $\tilde{x}_1 = d \neq p(\tilde{R}_1)$,

$$\varphi_1(\overset{\text{lie}}{R_1}, R_2, \ldots, R_n) \overset{\text{truth}}{\tilde{P}_1} \varphi_1(\overset{\text{truth}}{\tilde{R}_1}, R_2, \ldots, R_n),$$

in violation of *strategy-proofness*. Thus, $x_1 = \frac{\Omega}{n}$.

To prove that for each $i \in N\backslash\{1\}$, $x_i = \frac{\Omega}{n}$, suppose, on the contrary, that there is $i \in N\backslash\{1\}$ such that $x_{i1} < \frac{\Omega_1}{n}$. By *commodity-wise same-sidedness*, for each $j \in N\backslash\{1, i\}$, $x_{j1} \leq \frac{\Omega_1}{n}$, so that $\sum_N x_{j1} < \frac{\Omega_1}{n} + (n-1)\frac{\Omega_1}{n} = \Omega_1$, violating feasibility. Thus, for each $i \in N\backslash\{1\}$, $x_{i1} \geq \frac{\Omega_1}{n}$, and a similar argument shows that the latter inequality is indeed an equality. Now by *no-envy* applied to agents $2, \cdots, n$ and feasibility, it follows that for each $i \in N\backslash\{1\}$, $x_i = \frac{\Omega}{n}$.

Case 2: $c'_2 \leq \underline{p}_2(R_1)$.

An argument similar to that in Case 1 leads to a contradiction. We omit the details.

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